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2002 J. Phys. A: Math. Gen. 35 8005

(http://iopscience.iop.org/0305-4470/35/38/304)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 8005–8012 PII: S0305-4470(02)38623-2

Improved harmonic approximation and the 2D Ising model at $T \neq T_c$ and $h \neq 0$

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Received 25 June 2002
Published 12 September 2002
Online at stacks.iop.org/JPhysA/35/8005

Abstract
We propose a new method to determine the unknown parameter associated with a self-consistent harmonic approximation. We check the validity of our technique in the context of the sine-Gordon model. As a non-trivial application we consider the scaling regime of the 2D Ising model away from the critical point and in the presence of a magnetic field h. We derive an expression that relates the approximate correlation length ξ, $T - T_c$ and h.

PACS numbers: 11.10.−z, 11.10.Lm, 11.15.Tk

The so-called 'self-consistent harmonic approximation' (SCHA) is a non-perturbative technique that has been extensively employed in statistical mechanics [1, 2] and condensed matter physics [3–6] applications. Roughly speaking, it amounts to replacing an exact action S_{true} by a trial action S_{trial} that makes the problem tractable. Usually, S_{trial} is just a quadratic action that depends on a certain unknown parameter Ω that must be determined through some criterion such as the minimization of the free energy of the system. This approximation is intimately related to the 'Gaussian effective potential' [7, 8] in quantum field theories (QFTs), a variational approximation to the effective potential which uses a Gaussian wave functional depending on some mass parameter as the trial ground state. It also relies on a minimization principle often called the 'principle of minimal sensitivity' [9] to determine the additional parameter. In this work, we point out that in two-dimensional problems there is an alternative way of obtaining the quantity Ω. This method is based on conformal field theory (CFT) [10]. Moreover, we shall show that our method yields improved results with respect to the predictions of standard SCHA in the sine-Gordon (SG) model and allows us to give a new description of the off-critical 2D Ising model (2DIM). In the former, we exploit the existence of exact results [11, 12] to check the consistency of our proposal by obtaining a qualitatively good answer for the soliton mass. We then apply the same idea to the 2D Ising model at $T \neq T_c$ and $h \neq 0$, a non-integrable model in which very few quantitative results are known [13, 14].

We use the fermionic representation of the 2DIM. Since the standard SCHA is restricted to bosonic models, the new procedure is also an extension of the Gaussian approximation to fermionic 2D theories. Our main result is an algebraic equation which allows us to get the behaviour of the correlation length as a function of $T - T_c$ and h.

Let us stress that we are not introducing a new approximation but just a method to determine its parameter. As is well known, the SCHA is a non-controlled approximation, i.e. there is no perturbative parameter involved. It is then clear that the same criticism can be made of the present proposal.

We shall begin by depicting the main features of the standard SCHA. One starts from a partition function

$$\mathcal{Z}_{\text{true}} = \int \mathcal{D}\mu\, \mathrm{e}^{-S_{\text{true}}} \tag{1}$$

where $\mathcal{D}\mu$ is a generic integration measure and S_{true} is the exact action. An elementary manipulation leads to

$$\mathcal{Z}_{\text{true}} = \frac{\int \mathcal{D}\mu\, \mathrm{e}^{-(S_{\text{true}}-S_{\text{trial}})}\, \mathrm{e}^{-S_{\text{trial}}}}{\int \mathcal{D}\mu\, \mathrm{e}^{-S_{\text{trial}}}} \int \mathcal{D}\mu\, \mathrm{e}^{-S_{\text{trial}}} = \mathcal{Z}_{\text{trial}} \langle \mathrm{e}^{-(S_{\text{true}}-S_{\text{trial}})} \rangle_{\text{trial}} \tag{2}$$

for any trial action S_{trial}. Now, by means of the property

$$\langle \mathrm{e}^{-f} \rangle \geqslant \mathrm{e}^{-\langle f \rangle} \tag{3}$$

for f real, and taking the natural logarithm in equation (2), we obtain Feynman's inequality [15]

$$\ln \mathcal{Z}_{\text{true}} \geqslant \ln \mathcal{Z}_{\text{trial}} - \langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}}. \tag{4}$$

In general, S_{trial} depends on some parameters, which are fixed by minimizing the right-hand side of the last equation.

At this point, in order to illustrate the procedure, we shall consider the well-known SG model with the action

$$S_{\text{true}} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \varphi(p) \frac{F(p)}{2} \varphi(-p) + \int \mathrm{d}^2 x \frac{\alpha}{\beta^2} [1 - \cos(\beta\varphi)] \tag{5}$$

where $\varphi(p)$ is a scalar field and $F(p)$ is usually of the form $F(p) \sim p^2$. For simplicity, in this formula we have written the kinetic term in Fourier space but have kept the interaction term in coordinate space.

As the trial action one proposes a quadratic one,

$$S_{\text{trial}} = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \left[\varphi(p) \frac{F(p)}{2} \varphi(-p) + \frac{\Omega^2}{2} \varphi(p)\varphi(-p) \right] \tag{6}$$

where Ω is the trial parameter. In order to perform the standard minimization procedure, we first evaluate $\langle S_{\text{true}} - S_{\text{trial}} \rangle$. The result is

$$\langle S_{\text{true}} - S_{\text{trial}} \rangle_{\text{trial}} = \mathcal{V} \left[\frac{\alpha}{\beta^2} \left(1 - \mathrm{e}^{-\frac{1}{2}\beta^2 [I_1(\Omega) - I_1(\rho)]} \right) - \frac{\Omega^2}{2} I_1(\Omega) \right] \tag{7}$$

where

$$I_1(\Omega) = \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{[F(p) + \Omega^2]} \tag{8}$$

and ρ is a normal-ordering parameter [16].

Now inserting (7) in equation (4), and extremizing the rhs with respect to Ω, we finally obtain

$$\Omega^2 - \alpha\, \mathrm{e}^{-\beta^2/2(I_1(\Omega) - I_1(\rho))} = 0. \tag{9}$$

This gap equation allows us to extract a finite answer for Ω, depending on the mass parameter ρ (the difference $I_1(\Omega) - I_1(\rho)$ is finite). Note that the value of ρ is completely arbitrary, if one chooses it to be equal to the trial mass Ω, the solution to the equation is

$$\Omega^2 = \alpha. \tag{10}$$

The same result is obtained if instead of $\rho = \Omega$ one takes $\rho = \sqrt{\alpha}$.

Let us now present an alternative route to determine Ω. To this end, we will exploit a quantitative prediction of conformal invariance for 2D systems in the scaling regime, away from the critical point. Starting from the so-called 'c-theorem' [17], Cardy [18] showed that the value of the conformal anomaly c, which characterizes the model at the critical point, and the second moment of the energy–density correlator in the scaling regime of the non-critical theory are related by

$$\int d^2x |x|^2 \langle \varepsilon(x)\varepsilon(0)\rangle = \frac{c}{3\pi t^2 (2 - \Delta_\varepsilon)^2} \tag{11}$$

where ε is the energy–density operator, Δ_ε is its scaling dimension and $t \propto (T - T_c)$ is the coupling constant of the interaction term that takes the system away from criticality. The validity of this formula has been explicitly verified for several models [18, 19]. For the SG model, the energy–density operator is given by the cosine term, its conformal dimension is $\Delta_\varepsilon = \beta^2/4\pi$, t is the coupling constant α/β^2 and the associated free bosonic CFT has $c = 1$.

Now we claim that Ω can be determined in a completely different, not variational way, by enforcing the validity of the above conformal identity for the trial action. In other words, we will demand that the following equation holds:

$$\frac{\alpha^2}{\beta^4} \int d^2x |x|^2 \langle \cos\beta\varphi(x) \cos\beta\varphi(0)\rangle_{\text{trial}} = \frac{1}{3\pi \left(2 - \frac{\beta^2}{4\pi}\right)^2} \tag{12}$$

which is to be viewed as an equation for the mass parameter Ω. Of course, if one is interested in comparing the answer given by this formula with the SCHA result, when evaluating the left-hand side of (12) one must adopt a regularizing prescription equivalent to the normal ordering implemented in the SCHA calculation. A careful computation leads to the following gap equation:

$$\left(\frac{\Omega}{\rho}\right)^{2(2-u)} = \left(\frac{\alpha}{\rho^2}\right)^2 \frac{3}{32} \frac{2-u}{u^2} \tag{13}$$

where we have defined the variable $u = \beta^2/4\pi$ $(0 \leqslant u < 2)$ and ρ is the normal ordering parameter, as before. We see that, as in the standard SCHA equation (9), one has different answers for different choices of ρ, but in this case, the results obtained for the values $\sqrt{\alpha}$ and Ω are different. In any case, one gets a non-trivial dependence of Ω on β^2 in contrast with the SCHA. This is interesting if one recalls the physical meaning of mass gaps in the context of the SG model. Indeed, as is well known, Dashen, Hasslacher and Neveu (DHN) [11] have computed by semiclassical techniques the mass spectrum for the SG model. It consists of a soliton (associated with the fermion of the Thirring model) with the mass

$$M_{\text{sol}} = \frac{2-u}{\pi u}\sqrt{\alpha} \tag{14}$$

and a sequence of doublet bound states with masses

$$M_N = \frac{2(2-u)}{\pi u} \sin\left[N \frac{\pi u}{2(2-u)}\right] \sqrt{\alpha} \tag{15}$$

with $N = 1, 2, \ldots < (2-u)/u$. (From this last condition it is easy to see that in order to have N bound states one must have $u < 2/(N+1)$. As a consequence there is no bound state

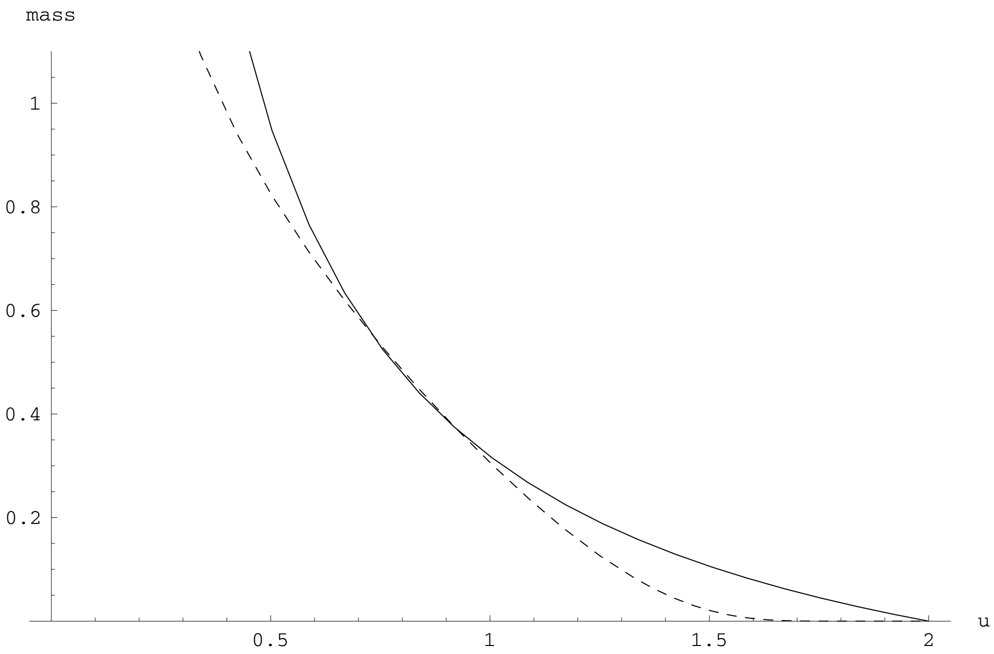

Figure 1. Masses in units of $\sqrt{\alpha}$ as functions of u. The full curve is $M_{\rm sol}/\sqrt{\alpha}$, whereas the dashed curve represents $\Omega/\sqrt{\alpha}$ as given by equation (13).

for $u > 1$.) More recently Zamolodchikov [12], by reinterpreting Bethe ansatz results, has given exact expressions for this spectrum. In particular, for the soliton his formula coincides very well with (14), except for u close to 2, where it predicts a divergence. For simplicity, here we compare our results with equation (14). The first thing to note is that the masses in the SGM spectrum also depend on u as our prediction given by equation (13). Thus, in this respect our proposal seems to be able to improve the standard Gaussian prediction for the SGM, at least qualitatively. In order to perform a more specific and quantitative discussion, let us compare equations (13) and (14) as functions of u. We set $\rho = \sqrt{\alpha}$, which corresponds to the prescription employed by DHN when deriving (14) and (15). The result is shown in figure 1 where one can observe a general qualitative analogy between both curves. In particular, for $0.7 \leqslant u \leqslant 1$ ($u = 1$ corresponds to the free fermion point of the Thirring model and to the Luther–Emery point in the backscattering model [20]) our prediction is in full agreement with the values of the soliton mass as computed by DHN. We want to stress that for $u = 1$ we get $\Omega/\sqrt{\alpha} = \sqrt{3/32} \approx 0.30$ whereas the value given by (14) is $1/\pi \approx 0.31$ (standard SCHA yields, of course, $\Omega/\sqrt{\alpha} = 1$).

Having checked the admissibility of our proposal in a model where exact results are known, it is now desirable to explore a non-trivial problem. Let us consider the 2D Ising model away from criticality ($T \neq T_{\rm c}$ and $h \neq 0$):

$$S = S_0 + \int {\rm d}^2x\, [t\epsilon(x) + h\sigma(x)] \tag{16}$$

where S_0 is the critical action, $t \propto (T - T_{\rm c})$, and $\epsilon(x)$ and $\sigma(x)$ are the energy–density and spin operators, respectively. We shall use the fermionic representation for the above action. Thus S_0 is a free massless Majorana action and $\epsilon \propto \bar{\Psi}\Psi$. On the other hand, the expression of $\sigma(x)$ in terms of the Majorana fields is more involved. Indeed, by means of a Jordan–Wigner transformation it can be written as an exponential of a fermionic bilinear. In analogy with the usual SCHA method, we propose the following quadratic trial action:

$$S_{\text{trial}} = S_0 + \Omega \int d^2 x \epsilon(x). \tag{17}$$

The conformal equation (11) for the present case takes the form

$$\int d^2 r r^2 [t^2 (2 - \Delta_\varepsilon)^2 \langle \varepsilon(r)\varepsilon(0)\rangle_{\text{trial}} + h^2 (2 - \Delta_\sigma)^2 \langle \sigma(r)\sigma(0)\rangle_{\text{trial}} + 2th(2 - \Delta_\varepsilon)(2 - \Delta_\sigma) \langle \varepsilon(r)\sigma(0)\rangle_{\text{trial}}] = \frac{1}{6\pi} \tag{18}$$

where we have set $c = 1/2$, which is the central charge corresponding to Majorana free fermions and $\Delta_\varepsilon = 1$ and $\Delta_\sigma = 1/8$ are the scaling dimensions of the corresponding operators. Now we have to evaluate the v.e.v.s in the trial theory. This will give us an equation for Ω as a function of t and h. The energy–energy and the energy–spin correlation functions have been computed by Hecht [24], whereas the spin–spin correlator can be found in the work of Wu *et al* [25]. As usual, one defines a correlation length $\xi = 1/4\Omega$ and considers the scaling limit given by $\xi \to \infty$, $r \to \infty$, with r/ξ fixed. The next step is to use the expressions of the correlators for $(r/\xi) \ll 1$ and perform the corresponding integrals. At this point, we have to take into account that the correlation functions are proportional to certain scaling functions $F_\pm(r/\xi)$, where the + and − signs correspond to the cases $\Omega > 0$ and $\Omega < 0$, respectively. In other words, the parameter Ω can be seen as defining a new 'effective' critical temperature, and the functions $F_\pm$ describe the scaling regime above and below this temperature. Since we are approximating a magnetic perturbation of the system it is clear that we must use the functions F_-. Thus, we obtain the following equation relating ξ, h and t:

$$t^2 (4\xi)^2 + C_1 h^2 (4\xi)^{15/4} + C2t|h|(4\xi)^{23/8} = 1 \tag{19}$$

where we have introduced the numerical constants $C_1 = 0.749\,661$ and $C_2 = 0.186\,966$. The absolute value of the magnetic field in the second term comes from the fact that $\langle \epsilon\sigma \rangle \propto \langle \sigma \rangle$ and the product $\langle \sigma \rangle h$ has to be positive since the magnetization and the magnetic field have the same orientation. For ξ fixed this equation gives a simple dependence of h as a function of t. Indeed, for $h > 0$ one has a slightly rotated semi-ellipse in the upper h–t-plane, and for $h < 0$ one has its reflection over the $t = 0$ axis.

An alternative form of equation (19) is obtained if one introduces the dimensionless combination $\chi = |h|^{-\frac{8}{15}}/4\xi_0$:

$$\left(\frac{\xi}{\xi_0}\right)^2 + C_1 \chi^{\frac{-15}{4}} \left(\frac{\xi}{\xi_0}\right)^{\frac{15}{4}} \pm C_2 \chi^{\frac{-15}{8}} \left(\frac{\xi}{\xi_0}\right)^{\frac{23}{8}} = 1. \tag{20}$$

The + and − signs in the third term of the left-hand side correspond to the cases $t > 0$ and $t < 0$, respectively. The action (16) defines a one-parameter family of field theories which can be labelled by χ. Moreover, the particle content of the model is expected to undergo drastic changes as a function of χ [13].

In order to check the consistency of the above equations we first consider the limits $h \to 0$ and $t \to 0$ separately. The first case corresponds to $\chi \to \infty$ and one immediately obtains $\xi = \xi_0$, as expected. In the second case, one has $\chi \to 0$ and then we get $\xi \sim |h|^{-\frac{8}{15}}$, which is in agreement with the exact result obtained in [21]. Let us mention that in this reference the constant of proportionality was exactly determined to be 4.4, whereas our approximate computation yields 3.7. Going back to the general case, we have solved equation (19) numerically for ξ as a function of χ for both $t > 0$ and $t < 0$. The results are plotted in figure 2. In the $t > 0$ case, the correlation length increases in a monotonic way from the zero

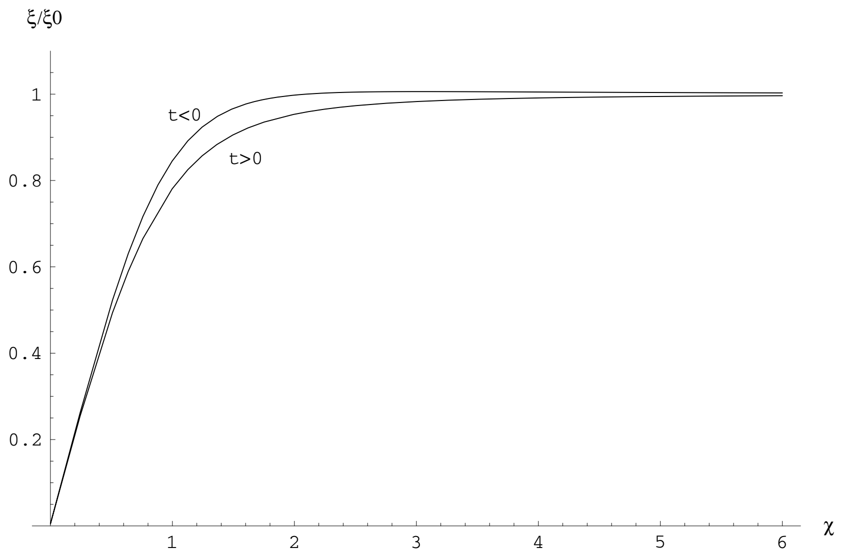

Figure 2. Correlation length in units of ξ_0 as a function of χ for both $t > 0$ and $t < 0$.

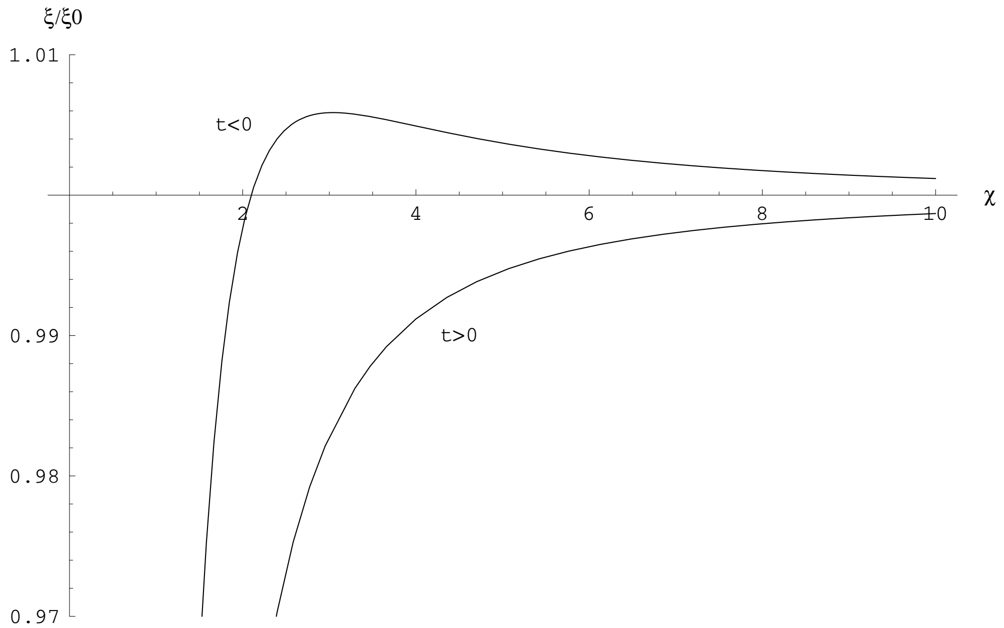

Figure 3. As figure 2, showing details of the behaviour of $\xi(\chi)$ for $t > 0$ and $t < 0$.

value and approaches the $h = 0$ value ξ_0 from below as $\chi \to \infty$. In the $t < 0$ case, although the behaviour of ξ seems very similar to the previous case, it presents a subtle difference shown in figure 3. For $\chi \approx 2$ the correlation function goes over the ξ_0 value, reaches a maximum and then tends to ξ_0 from above as $\chi \to \infty$. As this behaviour depends on the value of the constants C_1 and C_2 we do not know whether this is indeed a property of the Ising model or an artefact introduced by our approximation.

To conclude, we have reconsidered the well-known SCHA method in which a comparatively complex action is replaced by a simpler quadratic system depending on a mass parameter Ω which is usually determined through a variational calculation. Taking into account the $(1+1)$-dimensional case, we have proposed an alternative way of evaluating Ω. Our proposal is based on a consequence of Zamolodchikov's [17] c-theorem first derived by Cardy [18]. We have illustrated the idea by considering the SG model. We showed that for this model our method gives quite a good prediction for the behaviour of the soliton mass as a function of β^2 (see equations (13) and (14) and figure 1).

As a non-trivial application we have considered the 2D Ising model away from criticality ($T \neq T_c$ and $h \neq 0$). Starting from a continuum field theoretical description in terms of Majorana fermions, we proposed a quadratic trial action depending on a parameter Ω that defines an approximate correlation length ξ. Our main result is given by equation (19) (or its alternative form (20)) which allows us to determine the parameter Ω (i.e. ξ) in terms of the original physical parameters t and h.

It would be interesting to test our approach in other models such as the continuum version of the tricritical Ising model, which is described by the second model of the unitary minimal series [22, 23] with central charge $c = 7/10$.

Acknowledgments

This work was supported by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and Universidad Nacional de La Plata (UNLP), Argentina. We are grateful to Bernard Jancovici for fruitful e-mail exchanges and for drawing our attention to [12].

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